A Causal Semantics for the Edge Clique Cover Problem

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Abstract

We consider the task of causal structure learning over a set of measurement variables with no direct causal relations and whose dependencies are induced by unobserved latent variables. We call this the measurement dependence inducing latent (MeDIL) causal model, or MCM, framework. We show that this task can be framed in terms of the graph theoretical problem of finding edge clique covers, resulting in a simple algorithm for returning minimal MeDIL causal models (minMCMs). This algorithm is non-parametric, requiring no assumptions about linearity or Gaussianity. Furthermore, despite these rather weak and general assumptions, we are able to show that minimality in minMCMs implies three rather specific and interesting properties: first, minMCMs lower bound (i) the number of latent causal variables and (ii) the number of functional causal relations that are required to model a complex system at any level of granularity; second, a minMCM contains no causal links between the latent variables; and third, in contrast to factor analysis, a minMCM may require more latent than measurement variables.

The Edge Clique Cover (ECC) Problem

![Figure 1: example undirected graph](image)

The set of all maximal cliques for the above graph (i.e., the set consisting of all four 3-graphs) is an edge clique cover (ECC). Notice, however, that the clique formed by nodes A, B, and C can be removed and the three remaining cliques still cover all edges. These three cliques together form a minimal ECC. Finding a minimal ECC is more complicated and has higher complexity than finding the set of all maximal cliques.

Furthermore, there are two differrent kinds of minimality, clique-minimal and assignment-minimal:

![Figure 2(a): example undirected graph D(M) over variables M = \{M_1, ..., M_6\}](image)

![Figure 2(b): each C_i corresponds to a maximal clique in D(M) and each directed edge represents assignment—dashed red edges/vertices are redundant for clique-minimality while blue dotted edges/vertices are redundant for assignment-minimality;](image)

![Figure 2(c): 5 cliques C_i and 19 assignments for clique-minimal ECC over D(M)](image)

![Figure 2(d): 6 cliques C_i and 18 assignments for assignment-minimal ECC over D(M)](image)

MeDIL Causal Models

**Definition 1** (Measurement Dependence Inducing Latent Causal Model (MCM)). A graphical MCM is a DAG, given by the triple \( \mathcal{G} = (\mathcal{L}, \mathcal{M}, \mathcal{E}) \). \( \mathcal{L} \) and \( \mathcal{M} \) are disjoint sets of vertices, while \( \mathcal{E} \) is a set of directed edges between these vertices, subject to the following constraints:

1. all vertices in \( \mathcal{M} \) have in-degree of at least 1 and out-degree of 0
2. all vertices in \( \mathcal{L} \) have out-degree of at least 1
3. \( \mathcal{E} \) contains no cycles

To learn a minMCM for a given distribution of measurement variables, we represent the measurement variables \( \mathcal{M} \) as an undirected dependency graph upon which ECC finding algorithms can be applied. We denote this graph \( D(\mathcal{M}) \), and construct it by putting an undirected edge between two measurement variables if and only if they are unconditionally dependent. These dependencies can be learned from a set of samples via permutation-based hypothesis testing using non-linear measures of dependence, such as the Hilbert-Schmidt Independence Criterion (HSIC) or the distance correlation.

**Algorithm 1: constructing a minimal MeDIL causal model (minMCM)**

Input : \( D(\mathcal{M}) \) over the measurement variables \( \mathcal{M} \)
Output: vertex-minimal or assignment-minimal MCM \( \mathcal{G} \) over \( \mathcal{M} \)

1. initialize edgeless graph with a vertex for each \( M \in \mathcal{M} \);
2. use find_cm or find_am to get an edge clique cover of \( D(\mathcal{M}) \);
3. for each clique \( C \) in the cover do
   1. add vertex \( M \) with edges directed to each \( M \in C \);

Some interesting properties include:

- minMCMs lower bound (i) the number of latent causal variables and (ii) the number of functional causal relations that are required to induce the measurement variables
- minMCM has property that for all measurement variables, conditional independence relations are implied by unconditional independence relations (this is related to the Global Markov Property of a Markov Random Field)
- a minMCM contains no causal links between the latent variables
- in contrast to factor analysis or independent component analysis, a min-MCM may require more latent than measurement variables.

Conclusion

- some (in)dependence structures cannot be represented by a DAG over the corresponding variables but can be represented by an undirected dependency graph
- these structures are common for measurement variables, which are noisy copies or combinations of unobserved latents, e.g., in fMRI, calcium imaging, and psychiatric or econometric questionnaire data
- we propose the MeDIL causal model framework for use on such data and provide an algorithm for finding a minimal MCM for a given distribution of measurement variables.