A Transformational Characterization of Unconditionally Equivalent Bayesian Networks

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Outline

Introduction

Motivation

- Two DAGs $\mathcal{D}, \mathcal{D}'$ that share the same set of *d*-separation statements of the form $A \perp B \mid \emptyset$ are called *unconditionally equivalent*.
- Corresponding *unconditional equivalence classes (UECs)* of DAGs:
  - are easily estimated from data (e.g., nonzero entries of a covariance matrix),
  - are represented by simple undirected graphs,
  - but still contain a surprising amount of causal information.
- Hence, understanding UECs can help with:
  - (partial) causal discovery for otherwise prohibitively large structures
  - and speeding up existing causal discovery methods (e.g., GES).
- Previous work[1] focused on enumerating DAGs faithful to a given UEC.
- In contrast, we give novel characterizations and properties that facilitate exploring the space of UECs.

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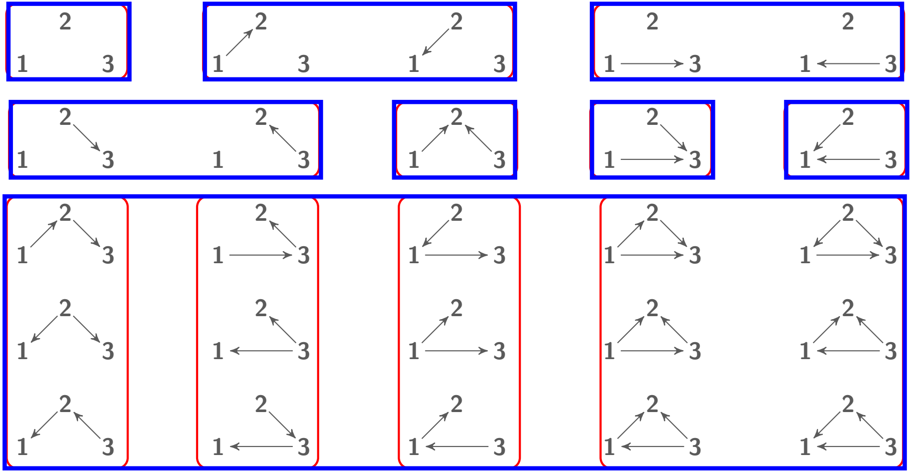
[1] J. Textor, A. Idelberger, and M. Liśkiewicz. "Learning from pairwise marginal independencies". In: *31st Conference on Uncertainty in Artificial Intelligence (UAI 2015)* (2015). arXiv:1508.00280 [cs.AI]

Introduction

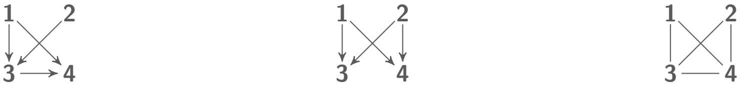
DAGs, MECs, and UECs example



Introduction

Unconditional dependence graphs (UDGs)

The *unconditional dependence graph (UDG)* of a DAG $\mathcal{D} = (V, E)$ is the undirected graph $\mathcal{U} = (V, \{\{v, w\} : v \not\perp_{\mathcal{D}} w;\ v, w \in V\})$.



(a) DAG $\mathcal{D}$ with d-separations $\mathcal{I} = \{1 \perp 2 \mid \emptyset,\ 2 \perp 4 \mid \{1, 3\}\}$

(b) DAG $\mathcal{D}'$ with d-separations $\mathcal{I} = \{1 \perp 2 \mid \emptyset,\ 3 \perp 4 \mid \{1, 2\}\}$

(c) UDG representing $\mathcal{I} = \{1 \perp 2\}$

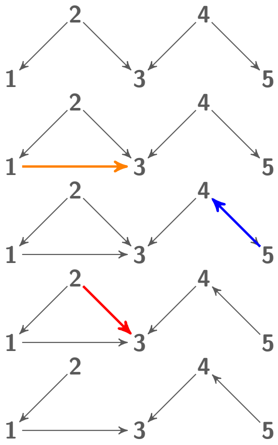
Figure: Unconditionally equivalent DAGs $\mathcal{D}, \mathcal{D}'$ with common unconditional d-separation statement $1 \perp 2$ and corresponding UDG missing edge $\{1, 2\}$

Transformational characterization for DAGs

Theorem (Transformational Characterization)

Let $\mathcal{D}$ and $\mathcal{D}'$ be two unconditionally equivalent DAGs. There exists a sequence of $|E^{D'} \setminus E^{D}|$ edge insertions, followed by $|\Delta(\mathcal{R}^{\mathcal{D},\mathcal{D}'}, \mathcal{R}^{\mathcal{D}',\mathcal{D}})|$ edge reversals, followed by $|E^{D} \setminus E^{D'}|$ edge deletions that transforms $\mathcal{D}$ into $\mathcal{D}'$ with the following properties:

1. *Each edge inserted or deleted in $\mathcal{D}$ is partially weakly covered or implied by transitivity.*
2. *Each edge reversed in $\mathcal{D}$ is weakly covered.*
3. *After each operation, the resulting $\mathcal{D}$ is a DAG and $\mathcal{U}^{\mathcal{D}} = \mathcal{U}^{\mathcal{D}'}$.*
4. *After all operations, $\mathcal{D} = \mathcal{D}'$.*



Transforming essential graphs (CPDAGs)

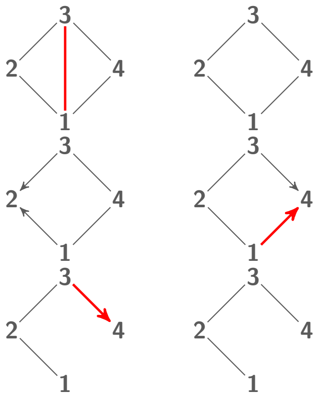
Theorem sketch and example

Theorem (11)

The edge $v \multimap w$ of an essential graph G is removable if and only if $\mathrm{mt}_{\mathcal{G}_{-v \multimap w}}(v) \subseteq \mathrm{mt}_{\mathcal{G}_{-v \multimap w}}(w)$.

Theorem (13)

Given an essential graph $\mathcal{G}$ and removable edge $v \multimap w$, define $T := \mathrm{ne}_{\mathcal{G}}(v) \cap \mathrm{ne}_{\mathcal{G}_{\mathrm{cc}(w)}}(w)$. Based on $|T|$ and whether $v \to w$ or $v - w$, we can determine if the PDAG $\mathcal{G}_{-v \multimap w}$ is complete and can enumerate all unconditionally equivalent completions otherwise.



Conclusion and future work

Conclusion: We derive novel characterizations of *unconditional equivalence classes (UECs)*, resulting in a new, more efficient way of moving around the space of DAGs.

Future work:

- Markov Chain Monte Carlo (MCMC) methods for learning UECs[2]
  - learn BIC-optimal UEC, MAP estimate, and full posterior $\pi(\mathcal{U} \mid \texttt{Data})$
- GES algorithms (including extension to MAGs)
  - first, take "big" steps between UECs in an uphill/forward phase
  - then, take "small" steps between CPDAGs (or PAGs) within the vastly reduced space of a single UEC in a downhill/backward phase
- generalizing to small instead of empty conditioning sets
  - more causal info than UECs; less computationally expensive than arbitrary conditioning sets

[2] D. Deligeorgaki, A. Markham, P. Misra, and L. Solus. "Combinatorial and algebraic perspectives on the marginal independence structure of Bayesian networks". In: (2022). arXiv:2210.00822 [stat.ME]